

Survival probabilities in time-dependent random walks

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We analyze the dynamics of random walks in which the jumping probabilities are periodic *time-dependent* functions. In particular, we determine the survival probability of biased walkers who are drifted towards an absorbing boundary. The typical lifetime of the walkers is found to decrease with an increment in the oscillation amplitude of the jumping probabilities. We discuss the applicability of the results in the context of complex adaptive systems.

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Random walk is one of the most ubiquitous concepts of statistical physics. In fact, it finds applications in virtually every area of physics (see e.g., Refs. [1–6], and references therein). Random walks in the presence of absorbing traps are much studied in recent years as models for absorbing-state phase transitions [7,8], polymer adsorption [9], granular segregation [10], and the spreading of an epidemic [11].

One of the main characteristics of a random walk in the presence of an absorbing boundary is the survival probability $S(t_0)$ of the walkers, the probability that a walker has not reached the absorption point before t_0 . In the present work we explore the properties of the survival probability of random walkers who experience a net drift towards an absorption point, and whose jumping probabilities are periodic *time dependent* functions, with a period T . The probability of stepping to the right at time t is given by

$$p(t) = \frac{1}{2} + F(t), \quad (1)$$

where $F(t)$ is a periodic function with a period T [$-1/2 < F(t) < 1/2$]. The mean of F (over one period) is given by

$$\langle F \rangle \equiv (1/T) \sum_{i=1}^T F(i) = -\epsilon_p, \quad (2)$$

where $\epsilon_p > 0$. We shall also consider cases in which the net drift is caused by a smaller step-size to the right (away from the absorption boundary) as compared to the step-size to the left.

In the unbiased case (in which case there is no net drift), the survival probability is well known to scale as an inverse power-law: $S(t) \propto t^{-1/2}$. When the walkers are drifted towards the absorbing point, the survival probability falls exponentially at asymptotically late-times $S(t) \propto t^{-3/2} \exp(-t/\tau_s)$, where the characteristic lifetime τ_s depends on the drift velocity (see e.g., Ref. [12]).

The problem of random walks with an absorption boundary has recently been extended to include a situation in which the jumping probabilities of the walkers are *time-dependent* [13]. The solution presented in [13] accounts for the binary (alternating) case, namely $p(t) = \frac{1}{2} - \epsilon_p + (-1)^t A$. Here A is the oscillations amplitude, and $\epsilon_p > 0$ represents a net drift towards the absorbing boundary (the period of such

binary oscillations is $T=2$). It was found [13] that the characteristic lifetime of the walkers τ_s depends on the amplitude A , in a nontrivial fashion. In particular, τ_s was shown to decrease monotonically with the increment of A . While this analysis provides a useful insight into the behavior of such time-dependent random walks, it is of one's interest to explore the general case, of time-dependent jumping probabilities with $T \neq 2$. This is the main goal of the present work.

In addition to the intrinsic interest in such time-dependent random walks, our work may find direct applications in many complex physical, biological, and economical systems. In fact, the main motivation for the introduction of time-dependent random walks in Ref. [13] was its applicability in the flourishing field of complex adaptive systems. In the well-studied model of the minority game (MG) [14], and its evolutionary version (EMG) [15] (see also [16–28]), it was found that the winning probabilities of the agents display a periodic behavior in time. This implies that the survival probabilities of the agents are well-described by a model of a periodic time-dependent random walk with an absorbing boundary [13]. The analytical model presented in Ref. [13] (with $T=2$) provides an elegant explanation for the intriguing phase-transition observed in the EMG [25] (from self-segregation to clustering, as the prize-to-fine ratio drops below some critical value). However, numerical studies of the EMG [27] have indicated that the period of the oscillations (in the winning probabilities of the agents) depends on the specific parameters of the system. Extending the analysis of Ref. [13] to the general case of time-dependent jumping probabilities with $T \neq 2$ is therefore highly motivated.

We consider a time dependent random walker who is drifted towards an absorption boundary, located at $-d$ (where $d > 0$). The drift towards the absorption point may be attributed to two distinct reasons: (i) A smaller (average) probability to take a step to the right (away from the absorption point). That is, $\langle p(t) \rangle = 1/2 - \epsilon_p$, with $\epsilon_p > 0$, and (ii) a smaller step size r to the right, $0 \leq r < 1$ (where the step size to the left is scaled to 1). This kind of drift is characterized by the parameter $\epsilon_r \equiv 1/(1+r) - \frac{1}{2}$. Below we explore the time-dependence of the survival probability $S(t)$, for a general periodic function $p(t)$.

In the time-independent case, it has been established that the probability $B(t)$ for a random walker to be located on the right-hand side at time t (in cases where there is *no* absorbing

boundary) is given by $B(t) \propto t^{-1/2} \exp[-t/\tau_B]$, with $\tau_B = \tau_s$ (see, e.g., Ref. [12]). Motivated by the equality $\tau_B = \tau_s$ in the time independent case, we will find the asymptotic form of τ_B in the general case of time-dependent jumping probabilities. We will establish the relation $\tau_B(T=2) \simeq \tau_s(T=2)$ analytically, and demonstrate numerically that $\tau_B(T \neq 2) \simeq \tau_s(T \neq 2)$.

We first consider the case in which the period T of the oscillations is a natural number. Let $\omega(t)$ be the number of right steps taken by the walker out of t steps. The walker is located at the right-hand side [$x(t) > 0$] if $\omega > \omega_c$, where

$$\omega_c = \frac{t-d}{1+r} \simeq t \left(\frac{1}{2} + \epsilon_r \right), \quad (3)$$

in the $t \gg d, T$ limit. Note that ω_c is purely a geometric quantity, and thus does not depend on p (accordingly, it does not depend on T , nor on ϵ_p).

At any given step, the probability to take a step to the right, $p(t)$ is an independent random variate. Hence, $\omega(t)$ (the total number of right steps) is the sum of t independent random variates. Thus, following the central limit theorem, the distribution of $\omega(t)$ approaches a normal distribution when t is large (the mean and the variance of this distribution is the sum of means and variances of p). For $t \gg T$ the distribution of ω can be very well approximated by a normal distribution with an average of $\mu_\omega(t) = t(\frac{1}{2} - \epsilon_p)$, and a variance $\sigma_\omega^2 = (t/T) \sum_{i=1}^T p_i(1-p_i) = t(1-4\langle F^2 \rangle)/4$, where

$$\langle F^2 \rangle \equiv (1/T) \sum_{i=1}^T F^2(i). \quad (4)$$

Hence, $B(t)$ can be approximated by this normal distribution, with the condition $\omega(t) > \omega_c(t)$:

$$\begin{aligned} B(t) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\omega_c}^{\infty} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx = \frac{1}{\sqrt{\pi}} \int_{\sqrt{t}/\tau_B}^{\infty} e^{-x^2} dx \\ &= \frac{1}{2} \operatorname{erfc}(\sqrt{t}/\tau_B), \end{aligned} \quad (5)$$

where

$$\tau_B = \frac{2\sigma^2 t}{(\omega_c - \mu)^2} = \frac{1-4\langle F^2 \rangle}{2(\epsilon_p + \epsilon_r)^2}, \quad (6)$$

and erfc is the complementary error function. The complementary error function can be approximated as $\operatorname{erfc}(\sqrt{t}/\tau_B) \simeq e^{-t/\tau_B} / \sqrt{\pi t / \tau_B}$ at asymptotically large times. One therefore finds

$$B(t) \propto t^{-(1/2)} e^{-t/\tau_B}, \quad (7)$$

where τ_B is given by Eq. (6).

The model of binary time-dependent jumping probabilities [$F(t) = -\epsilon_p + A(-1)^t$] was solved *exactly* in Ref. [13], the expression for τ_s is given by Eq. (16) of [13]. Expanding this expression for small ϵ_p and ϵ_r , one finds

$$\tau_s(T=2) \simeq \frac{1-4A^2}{2(\epsilon_p + \epsilon_r)^2}. \quad (8)$$

This agrees with Eq. (6) in the limit $\epsilon_p, \epsilon_r \ll 1$ (or, equivalently, $\tau_s \gg 1$). One therefore finds $\tau_s(T=2) \simeq \tau_B(T=2)$ in the $\tau_s \gg 1$ limit.

The generalization of the above analysis for an arbitrary period T is straightforward. If $T = m/n$ is a rational number, then $F(t)$ could be replaced by an equivalent function with a period m , which is a natural number. If T is an irrational, then the all range of values of $F(0 < t < T)$ is sampled (over many periods of the oscillations). In this case, one should replace Eqs. (2) and (4) by

$$\langle F \rangle = \frac{1}{T} \int_0^T F(t) dt = -\epsilon_p \quad (9)$$

and

$$\langle F^2 \rangle = \frac{1}{T} \int_0^T F^2(t) dt. \quad (10)$$

In order to confirm the analytical predictions [and most importantly, the relation $\tau_s(T \neq 2) \simeq \tau_B(T \neq 2)$], we perform numerical simulations of (discrete) random walks. As we have proved analytically that our approximation holds true in the binary case with $T=2$, we shall compare the survival probabilities $S(t)$ in the $T=2$ and $T \neq 2$ cases. By finding the relation between $\tau_s(T \neq 2)$ and $\tau_s(T=2)$, and showing that it is similar to the analytic relation between $\tau_B(T \neq 2)$ and $\tau_B(T=2)$, we shall establish the result $\tau_s = \tau_B$ in the generic case (with $T \neq 2$).

We consider numerically two different periodic behaviors for the jumping probabilities. In the first case, $p(t)$ is given by an harmonic function, while in the second case it is given by a square function. In the harmonic case, $p(t)$ is of the form

$$p(t) = \frac{1}{2} - \epsilon_p - A \cos\left(\frac{2\pi t}{T}\right); T > 1. \quad (11)$$

(We emphasize that t is a *discrete* variable.) In this case one finds [29] $\langle F \rangle = -\epsilon_p$, and

$$\langle F^2 \rangle = \begin{cases} \epsilon_p^2 + A^2; & T=2 \\ \epsilon_p^2 + (A/\sqrt{2})^2; & T \neq 2. \end{cases} \quad (12)$$

This implies that for $T \neq 2$, τ_B is *independent* of the period T of the oscillations. Moreover, we find that $\tau_B(T \neq 2, A/\sqrt{2}) = \tau_B(T=2, A)$.

Figure 1 depicts $t(S=10^{-5})$, the step number in which the survival probability has fallen to $S=10^{-5}$, as a function of the oscillation period T (and for three different values of the oscillation amplitude A). As predicted by Eq. (12), the survival probabilities are almost independent of the period T , except in the unique case of binary oscillations with $T=2$. Moreover, $S(t; T=2) < S(t; T \neq 2)$, in agreement with Eqs. (6) and (12). In addition, the numerical simulations indicate that

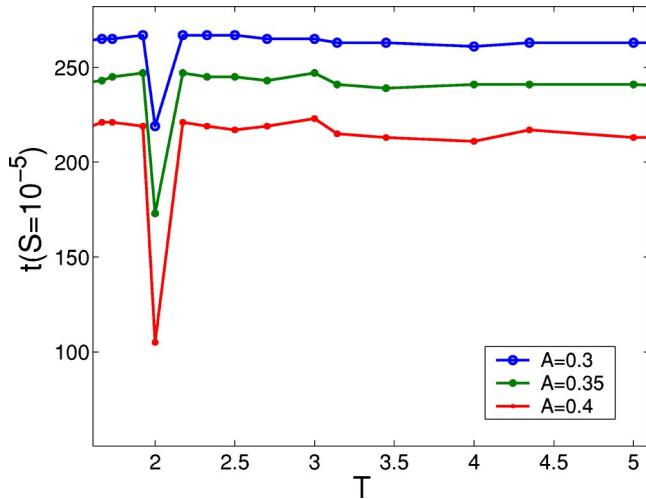


FIG. 1. The step number for which the survival probability has fallen to $S=10^{-5}$, as a function of the oscillation period T , and for three different values of A . The jumping probabilities $p(t)$ are characterized by harmonic oscillations, and are given by Eq. (11). We use $N=10^8$ walkers, $\epsilon_p=0.1$, $r=1$, and $d=2$. All quantities are dimensionless. $S(t)$ is found to be independent of the oscillation period T , except in the unique case of binary oscillations with $T=2$ [in which case $S(t)$ is smaller, in agreement with the analytical prediction.] In addition, $S(t)$ decreases monotonically with the increment of the oscillations amplitude A .

the survival probability decreases monotonically with an increment in the oscillation amplitude A , in agreement with the analytical prediction Eq. (6).

In Fig. 2 we confirm the relation $S(t, T \neq 2, A) = S(t, T = 2, \sqrt{2}A)$ in the harmonic case. By doing so we confirm that $\tau_s(T \neq 2, A/\sqrt{2}) = \tau_s(T = 2, A)$, and thus that $\tau_s(T \neq 2) \approx \tau_B(T \neq 2)$. The figure displays the step number $t(S=10^{-6})$ for which the survival probabilities have fallen to 10^{-6} . It is clear

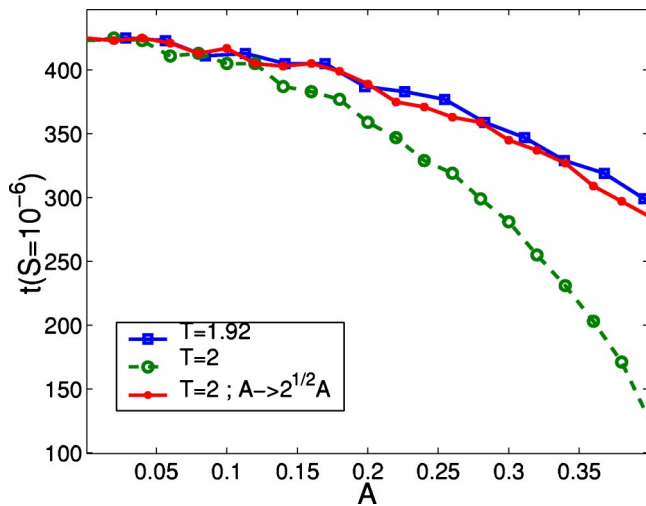


FIG. 2. The step number for which $S=10^{-6}$, as a function of the oscillations amplitude A , and for two different values of the period T . The jumping probabilities $p(t)$ are characterized by harmonic oscillations, and are given by Eq. (11). The results confirm the analytical prediction according to which $S(t, T \neq 2, A) = S(t, T = 2, \sqrt{2}A)$. The parameters used are the same as in Fig. 1.

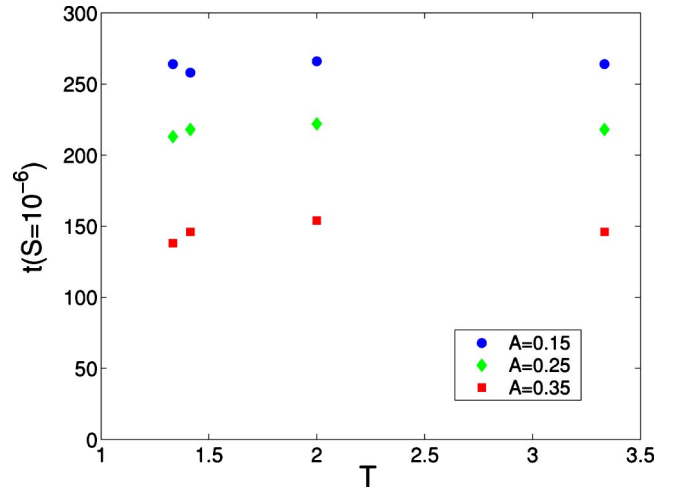


FIG. 3. The step number for which the survival probability has fallen to $S=10^{-6}$, as a function of the oscillation period T , and for three different values of A . The jumping probability is given by a square wave of the form $p(t) = \frac{1}{2} - \epsilon_p + Ax(t)$, where $x(t) = +1$ for $2nT < t \leq (2n+1)T$, and $x(t) = -1$ otherwise, where n is an integer. We use $N=10^9$ walkers, $\epsilon_p=0.1$, $r=0.9$, and $d=2$. We find that in this case, $S(t)$ is independent of the oscillation period T , in agreement with the analytical results.

that $t(S=10^{-6})$ for $T=2$ agrees with the corresponding value in the $T \neq 2$ case, provided one uses the transformation $A \rightarrow \sqrt{2}A$.

We further check the validity of the analytical results for cases in which the jumping probabilities are given by a square wave of amplitude A and a mean value $\langle F \rangle = -\epsilon_p$, namely $p(t) = \frac{1}{2} - \epsilon_p + Ax(t)$, where $x(t) = +1$ for $2nT < t \leq (2n+1)T$, and $x(t) = -1$ otherwise, where n is some integer. In this case one finds $\langle F^2 \rangle = \epsilon_p^2 + A^2$ for any value of T . This implies that $\tau_B(T)$ should be independent of the period T of the oscillations. Figure 3 depicts $t(S=10^{-6})$, the step number for which the survival probability has fallen to $S=10^{-6}$, as a function of the period T . We display results for three different values of the oscillation amplitude A . The numerical results confirm the finding according to which $S(t; T)$ is independent of the value of T , confirming the relation $\tau_s \approx \tau_B$. Furthermore, the survival probability $S(t; T)$ decreases with increasing amplitude A , in agreement with Eq. (6).

In summary, in this paper we have studied the survival probabilities of biased (drifted) random walkers, whose jumping probabilities are *time-dependent*. This is an extension of the binary case (with $T=2$), studied in Ref. [13]. The long-time asymptotic survival probability is dominated by an exponential fall $S(t) \propto \exp[-t/\tau_s]$. We have found a simple approximation for τ_s [see Eq. (6)], which solely depends on the first and the second moments of the oscillations ($\langle F \rangle$ and $\langle F^2 \rangle$), and on the step-size drift parameter ϵ_r .

Our analytical results imply that the characteristic lifetime of the walkers τ_s decreases with increasing $\langle F^2 \rangle$. The larger are the temporal-oscillations in the jumping probabilities $p(t)$, the smaller is the survival probability. This result generalizes the one derived in Ref. [13] for the binary case (with $T=2$).

The qualitative nature of the anticorrelation between the survival probability $S(t)$ and the oscillation amplitude $\langle F^2 \rangle$ (in the biased case) is rather simple. It is a direct consequence of the positive correlation between $\sigma_\omega(t)$ (the dispersion in the number of right-steps taken out of a total of t steps) and $S(t)$. In the presence of a net drift towards the absorption boundary, μ_ω (the average value of ω) is smaller than ω_c . A large $\sigma_\omega(t)$ is therefore required in order to survive, that is, in order to have $\omega > \omega_c$ at asymptotically late times [30].

Finally, the present analysis provides a direct explanation for the underlying mechanism responsible for the intriguing phase-transition observed in the evolutionary minority game [25,15]. In this model, the winning probabilities of the agents $p(t)$ were shown to display temporal oscillations [27]. The amplitude and period of these oscillations depend on the

various parameters of the model (such as the price-to-fine ratio). In [13] a toy-model of time-dependent jumping probabilities with a period of $T=2$ was used to reveal the physics behind the dynamical phase-transition. In the present work, we have shown that the anticorrelation between the characteristic lifetime τ_s and the oscillations amplitude $\langle F^2 \rangle$ is a generic feature, *independent* of the period T of the oscillations. Thus, our analysis (for generic values of T) lends support for the general applicability of the conclusions presented in [13].

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- [30] We note that in the unbiased case (with no drift), one finds $\mu_\omega > \omega_c$. A small $\sigma_\omega(t)$ therefore implies large survival probability. Thus, $\langle F^2 \rangle$ is positively correlated with $S(t)$ in the unbiased case, in agreement with the results of [13] for the binary case.